Thus it can be shown that

$$\lambda_n b = \lambda_n c \ \alpha = \alpha\delta_n \tag{22a}$$

$$\lambda_n a = \lambda_n c \ \epsilon\alpha = \alpha\epsilon\delta_n \tag{22b}$$

where $\delta_n = \lambda_n c$, the roots of $J_1(\delta_n) = 0$.

If we further define $R_a^* = kbR_a$ and $u = r/b$, then we can show that

$$R_a^* = \frac{(2/\pi)}{(1-\epsilon^2)\int_\epsilon^1 uf(u)\,du} \sum_{n=1}^{\infty} \frac{J_1(\alpha\delta_n)\left\{1-\epsilon\dfrac{J_1(\alpha\epsilon\delta_n)}{J_1(\alpha\delta_n)}\right\}}{\delta_n^2 J_0^2(\delta_n)} \int_\epsilon^1 uf(u) J_0(\alpha\delta_n u)\,du \tag{23}$$

is the general expression for the thermal constriction resistance due to an annular contact area with an arbitrary flux distribution over the contact area. R_a^* often is called the constriction parameter and is defined as ψ_a.

Equation (23) is valid for any axially symmetric flux distribution and can be evaluated analytically or numerically. For the case of a uniform heat flux, $f(u) = 1$, we obtain

$$\int_\epsilon^1 uf(u)\,du = \tfrac{1}{2}(1-\epsilon^2) \tag{24}$$

$$\int_\epsilon^1 uf(u) J_0(\alpha\delta_n u)\,du = \frac{1}{(\alpha\delta_n)^2}\{(\alpha\delta_n) J_1(\alpha\delta_n) - (\alpha\epsilon\delta_n) J_1(\alpha\epsilon\delta_n)\} \tag{25}$$

With Eqs. (24) and (25), Eq. (23) becomes

$$\psi_a = \frac{(4/\pi)}{(1-\epsilon^2)^2} \sum_{n=1}^{\infty} \frac{J_1^2(\alpha\delta_n)\left\{1-\epsilon\dfrac{J_1(\alpha\epsilon\delta_n)}{J_1(\alpha\delta_n)}\right\}^2}{\delta_n^3 J_0^2(\delta_n)} \tag{26}$$

in agreement with the analysis of Yip.[4] In his dissertation, Yip presents a plot of ψ_a for various values of ϵ and α. It can be seen that the general expression developed here reduces to the general expression valid for a circular contact area[1] when ϵ is set equal to zero.

Conclusions

A general annular constriction parameter valid for axially symmetric flux distributions has been derived from first principles. In the limiting case of a circular contact, the general expression developed here reduces to one developed previously. The annular constriction parameter for the case of uniform heat flux is seen to be a particular case of the general case, and it was obtained with great ease from the general expression.

References

[1] Yovanovich, M.M., "General Expressions for Constriction Resistances Due to Arbitrary Flux Distributions At Non-Symmetric, Coaxial Contacts," AIAA Paper 75-188, Jan. 1975, Pasadena, Calif.

[2] Abramowitz, M. and Stegun, I., *Handbook of Mathematical Functions*, Dover, New York, 1965.

[3] Gradshteyn, I.S. and Ryzhik, I.M., *Table of Integrals, Series, and Products*, Academic Press, New York, 1965.

[4] Yip, F.C., Ph.D. Thesis, 1969, Dept. of Mechanical Engineering, University of Calgary, Calgary, Alberta.

Notes on Finite Elements for Nearly Incompressible Materials

T. H. H. Pian* and S. W. Lee†

Massachusetts Institute of Technology, Cambridge, Mass.

Introduction

In applying conventional finite element methods to incompressible materials, the resulting element stiffness matrix will be infinite. Alternative formulations must be provided whereby the mean stresses in the elements are introduced as unknowns in addition to the nodal displacements. Herrmann[1] and Key[2] introduced such remedies for the assumed displacement method, while Tong[3] and Lee[4] provided corresponding schemes for the assumed stress hybrid model. For nearly incompressible materials it is also known that the conventional finite element models based on the assumed displacement approach do not give reliable solutions, unless the mean stresses are retained as additional variables. Nagtegaal et al.,[5] in explaining the failure of conventional finite element solutions in the fully plastic range by the tangent stiffness approach, have pointed out the severe kinematic constraints on the modes of deformation introduced by the zero dilatation condition. They have demonstrated that a plane strain element made of four constant strain triangles forming two diagonals can give satisfactory solutions for the fully plastic range.

The purpose of this Note is to point out that the incompressibility constraints imposed on the assumed stress hybrid elements are much less severe. Hence, solutions for materials very close to incompressibility, can be obtained with sufficient accuracy by such elements using the conventional method which contains only nodal displacements as unknowns. This Note will also point out the situations for which the formulations by Key can also be converted to the conventional matrix displacement methods for nearly incompressible materials. In the following developments, only isotropic materials will be considered and the effect of body forces will be excluded for the sake of simplicity.

Incompressibility

The pointwise incompressibility condition i.e., the condition of vanishing dilatational strain, provides certain constraints to the assumed displacement finite element formulation. As an example the displacements for a four node rectangular plane strain element can be assumed as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \tag{1a}$$

$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy \tag{1b}$$

However, to satisfy the condition $\epsilon_x + \epsilon_y = (\partial u/\partial x) + (\partial v/\partial y) = 0$, it is necessary that

$$\alpha_2 + \alpha_7 = 0 \tag{2a}$$

$$\alpha_4 = 0 \tag{2b}$$

$$\alpha_8 = 0 \tag{2c}$$

These three conditions of constraint will limit the element motion to only two possible deformation modes which, of course, cannot represent more general state of strain.

Received Jan. 29, 1976. This work was sponsored by the Air Force Office of Scientific Research under Contract F44620-72-C-0018.

Index category: Structural Static Analysis.

*Professor of Aeronautics and Astronautics.

†Research Assistant, Department of Aeronautics and Astronautics.

Assumed Stress Hybrid Element

The assumed stress hybrid element[6,7] is based on independently assumed equilibrating stresses in the elements and compatible displacements along the boundary of the elements. The variational principle to be used is the stationary condition of the following functional

$$\pi_{mc}=\sum_n\{\int_{\partial V_n}T_iu_idS-\int_{V_n}\frac{1}{2}S_{ijk\ell}\sigma_{ij}\sigma_{k\ell}dV-\int_{S_{\sigma_n}}T_iu_idS\} \quad (3)$$

where

σ_{ij} = stress tensor
T_i = boundary traction
$S_{ijk\ell}$ = elastic compliance tensor
u_i = boundary displacement
V_n = volume of the nth element
∂V_n = entire boundary of V_n
S_{σ_n} = portion of the boundary of V_n where the surface tractions are prescribed

In this finite element formulation, the stresses $\boldsymbol{\sigma}$ and boundary displacements $\mathbf{u}$ are independently expressed in terms of stress parameters $\boldsymbol{\beta}$ and nodal displacements $\mathbf{q}$. Equation (3) then can be expressed in terms of $\boldsymbol{\beta}$ and $\mathbf{q}$ in the form of

$$\pi_{mc}=\sum_n(\boldsymbol{\beta}^T\mathbf{G}\mathbf{q}-\frac{1}{2}\boldsymbol{\beta}^T\mathbf{H}\boldsymbol{\beta}-\mathbf{q}^T\mathbf{Q}) \quad (4)$$

But the stationary condition enables us to express the stress parameters $\boldsymbol{\beta}$ in terms of $\mathbf{q}$, and π_{mc} can be written as

$$\pi_{mc}=\Sigma[\frac{1}{2}\mathbf{q}^T\mathbf{k}\mathbf{q}-\mathbf{q}^T\mathbf{Q}] \quad (5)$$

where

$$K=G^T\mathbf{H}^{-1}\mathbf{G}=\text{element stiffness matrix} \quad (6)$$

For incompressible materials the $\mathbf{H}$ matrix becomes singular because of the linear dependency of those columns corresponding to the constant normal stress terms. It can be shown in the following that for certain arrangements of the stress parameters the element stiffness matrix $\mathbf{k}$ can be written in the form of

$$\mathbf{k}=\mathbf{k}_1+(\mathbf{k}_2/\epsilon c) \quad (7)$$

where ϵ is equal to (1-2ν) and goes to zero as the Poisson's ratio approaches 0.5, and c is a constant. The constant terms of the assumed normal stress components may be written as

$$\sigma_x=\beta_{01}+\alpha+\cdots \quad (8)$$

$$\sigma_y=\beta_{02}+\alpha+\cdots$$

$$\sigma_z=-(\beta_{01}+\beta_{02})+\alpha+\cdots$$

The stress parameters thus can be expressed as

$$\boldsymbol{\beta}=\begin{Bmatrix}\beta_0\\ \alpha\end{Bmatrix} \quad (9)$$

$$G=\begin{Bmatrix}G_0\\ G_\alpha\end{Bmatrix} \quad (10)$$

$$H=\begin{bmatrix}H_0 & \epsilon H_{\beta\alpha}\\ \epsilon H_{\beta\alpha}^T & \epsilon a\end{bmatrix} \quad (11)$$

It can be shown that the factor ϵ only appears in the last column and row in $\mathbf{H}$. Equation (4) can then be written as

$$\pi_{mc}=\sum_n[\beta_O{}^T\mathbf{G}_O\mathbf{q}+\alpha\mathbf{G}_\alpha\mathbf{q}-\frac{1}{2}\beta_O{}^T\mathbf{H}_O\beta_O$$
$$-\epsilon\beta_O{}^T\mathbf{H}_{\beta\alpha}\alpha-\epsilon\frac{a}{2}\alpha^2-\mathbf{q}^T\mathbf{Q}] \quad (12)$$

Again the stationary condition enables us to express β_O and α in terms of $\mathbf{q}$ and π_{mc} can be expressed as

$$\pi_{mc}=\sum_n[\frac{1}{2}\mathbf{q}^T(\mathbf{k}_1+\frac{\mathbf{k}_2}{\epsilon(a-\epsilon\mathbf{H}_{\beta\alpha}{}^T\mathbf{H}_0{}^{-1}\mathbf{H}_{\beta\alpha})}\mathbf{q}-\mathbf{q}^T\mathbf{Q}] \quad (13)$$

where

$$\mathbf{k}_1=\mathbf{G}_O{}^T\mathbf{H}_O{}^{-1}\mathbf{G}_O+\frac{1}{(a-\epsilon\mathbf{H}_{\beta\alpha}{}^T\mathbf{H}_O{}^{-1}\mathbf{H}_{\beta\alpha})}$$
$$[-(\mathbf{G}_\alpha{}^T\mathbf{H}_{\beta\alpha}{}^T\mathbf{H}_O{}^{-1}\mathbf{G}_O+\mathbf{G}_O{}^T\mathbf{H}_O{}^{-1}\mathbf{H}_{\beta\alpha}\mathbf{G}_\alpha)$$
$$+\epsilon\mathbf{G}_O{}^T\mathbf{H}_O{}^{-1}\mathbf{H}_{\beta\alpha}\mathbf{H}_{\beta\alpha}{}^T\mathbf{H}_O{}^{-1}\mathbf{G}_O] \quad (14)$$

and

$$\mathbf{k}_2=\mathbf{G}_\alpha{}^T\mathbf{G}_\alpha \quad (15)$$

This verifies Eq. (7). As ν approaches 0.5, it is necessary that

$$\mathbf{G}_\alpha\mathbf{q}=0 \quad (16)$$

For the assumed stress hybrid element the incompressibility condition thus always leads to only one constraint, which in fact, specifies that the volume of the element be maintained constant. It should be remarked that the element stiffness matrix obtained in this manner is identical to that by the ordinary arrangement that the three normal stresses be β_1, β_2, and β_3, respectively.

Formulation by Key's Principle

In the functional given by Key [2], the strain energy in the variational functional is separated into two parts, the deviatoric part expressed in terms of displacements and the spherical parts, in terms of displacements and stresses, i.e.

$$\pi_{mR}=\sum_n[\int_{V_n}A(\epsilon_{ij}')dV+\int_{V_n}(\sigma_m\epsilon_{kk}-\frac{1}{2\kappa}\sigma_m{}^2)dV-\int_{S_{\sigma_n}}T_iu_idS] \quad (17)$$

where

$A(\epsilon_{ij}')$ = deviatoric strain energy in terms of deviatoric strain ϵ_{ij}'
ϵ_{kk} = dilatational strain
σ_m = mean stress
κ = bulk modulus $=E/3(1-2\nu)$

Both ϵ_{ij}' and ϵ_{kk} are, of course, expressed in terms of displacement components.

In the finite element formulation, σ_m and $\mathbf{u}$ are expressed in terms of stress parameters $\boldsymbol{\beta}$ and nodal displacements $\mathbf{q}$ and π_{mR} can be expressed in the form of

$$\pi_{mR}=\sum_n[\frac{1}{2}\mathbf{q}^T\mathbf{k}_1\mathbf{q}+\boldsymbol{\beta}^T\mathbf{G}_*\mathbf{q}-\frac{1}{2\kappa}\boldsymbol{\beta}^T\mathbf{H}_*\boldsymbol{\beta}-\mathbf{q}^T\mathbf{Q}] \quad (18)$$

Here again $\boldsymbol{\beta}$ can be expressed in terms of $\mathbf{q}$ in the element level and

$$\pi_{mR}=\sum_n[\frac{1}{2}\mathbf{q}^T(\mathbf{k}_1+\kappa\mathbf{k}_2)\mathbf{q}-\mathbf{q}^T\mathbf{Q}] \quad (19)$$

where

$$\mathbf{k}_2=\mathbf{G}_*{}^T\mathbf{H}^{-1}\mathbf{G}_* \quad (20)$$

For the case of incompressibility where κ becomes infinite it is necessary that

$$\mathbf{G}_*\mathbf{q}=0 \quad (21)$$

Table 1 Solutions of the displacements at the unsupported corner of a square-shaped plane strain panel (10^{-3} in.)

Element	Hybrid stress model		Key's principle		Displacement model bi-linear		Displacement model four subtriangles	
ν \ mesh	1×1	2×2 4×4	1×1	2×2	1×1	2×2	1×1	2×2
0.33	0.191	0.185	0.201	0.186	0.178	0.182	0.170	0.183
0.40	0.178	0.172	0.194	0.173	0.150	0.164	0.161	0.167
0.45	0.164	0.160	0.188	0.160	0.110	0.139	0.114	0.152
0.48	0.153	0.151	0.183	0.150	0.634×10^{-1}	0.106	0.130	0.142
0.4999	0.145	0.144	0.180	0.143	0.479×10^{-3}	0.189×10^{-2}	0.120	0.133
0.49995	0.145	0.144	0.180	0.143	0.240×10^{-3}	0.953×10^{-3}	0.120	0.133

These are again the incompressibility constraints. It is seen that the number of constraints is equal to the number of stress parameters β. To restrict the number of constraints to a minimum it is desirable to assume constant mean stress σ_m for the entire element. In that case the constraint condition is again the maintaining of constant volume of the element. However, if the mean stress distribution is assumed linear in x and y there will exist three kinematic constraints which are exactly the same as Eq. (2). The assumption of constant mean stress for the entire element is, however, unreasonable if the deviatoric strains contain higher order terms.

Example Problem

A plane strain problem was solved and double precision was used to minimize the round off error. The structure is 1″ × 1″ square planform with one horizontal edge clamped and the neighboring edge free to slide. The other horizontal edge is acted on by a uniform tensile stress of 4000 psi. Different rectangular elements used are: 1) four mode assumed stress hybrid element with seven stress parameters; 2) four node element with bilinear displacement assumption; 3) an element made of four constant strain triangles forming two diagonals, and 4) four node element derived by Key's principle with constant mean stress.

It should be noted that the scheme of subelement construction for the third element above is not applicable to other types of elements such as axisymmetric solids and three-dimensional solids. Solutions for the vertical displacement at the unsupported corner are given in Table 1. Here the assumed stress hybrid model using 2×2 and 4×4 meshes are identical hence they can be considered as the reference solutions of the problem.

It is seen that the assumed stress hybrid model yields the most accurate solutions for all values of ν, although the element with four subtriangles and the one by Key's principle also led to reasonable solutions for ν approaching 0.5. On the other hand, in the case of element based on bilinear displacement, the boundary conditions of the present problem will prevent any deformation on account of the severe constraints due to the incompressibility condition.

References

[1]Herrmann, L. R., "Elasticity Equations for Incompressible and Nearly Incompressible Materials by a Variational Theorem," *AIAA Journal*, Vol. 3, 1965, pp. 1896-1900.

[2]Key, S. W., "A Variational Principle for Incompressible and Nearly Incompressible Anisotropic Elasticity," *International Journal of Solids and Structures*, Vol. 5, 1969, pp. 951-964.

[3]Tong, P., "An Assumed Stress Hybrid Finite Element Method for an Incompressible and Near-Incompressible Material," *International Journal of Solids and Structures*, Vol. 5, 1969, pp. 455-461.

[4]Lee, S. W., "An Assumed Stress Hybrid Finite Element for Three Dimensional Elastic Structural Analysis," M.I.T. ASRL TR 170-3, also AFOSR TR 75-0087, May 1974.

[5]Nagtegaal, J. C., Parks, D. M. and Rice, J. R., "On Numerically Accurate Finite Element Solutions in the Fully Plastic Range," *Computer Methods in Applied Mechanics and Engineering*, Vol. 4, 1974, pp. 153-157.

[6]Pian, T. H. H., "Derivation of Element Stiffness Matrices by Assumed Stress Distributions," *AIAA Journal*, Vol. 2, 1964, pp. 1333-1336.

[7]Pian, T. H. H. and Tong, P., "Finite Element Methods in Continuum Mechanics," *Advances in Applied Mechanics*, Vol. 12, Edited by C. S. Yih, Academic Press, 1972, pp. 1-58.

Technical Comments

Comment on "Extended Integral Equation Method for Transonic Flows"

Helge Nørstrud*
Norwegian Institute of Technology, University of Trondheim, Norway

Introduction

NIXON[1] has reported on a method of analysis for calculating transonic flows which he describes as the extended integral equation method. This method, he states, results in a considerable improvement in the accuracy with respect to the "standard" integral equation which, on the other hand, yields unsatisfactory results. To visualize the gain in accuracy Nixon compares one of his obtained solutions with a "standard" integral equation solution[2,3]. The purpose of the present Note is threefold: to show that the comparison presented by Nixon is erroneous and, hence, misleading; to bring some results which qualify the validity of the approximation made in the evaluation of the subject field integral; and to comment on an earlier criticism[4] of related nature.

Received Nov. 17, 1975.
Index category: Subsonic and Transonic Flow.

*Associate Professor, Division of Aero- and Gas Dynamics. Associate Fellow, AIAA.

Analysis

The problem under discussion concerns the solutions of the nonlinear potential equation which describes the transonic small-disturbance flow past a slender profile. The integral equation method formulates the solution to this partial differential equation as an integral equation in a reduced space of the physical coordinates. Similarity solutions are then obtained, but the analysis requires a knowledge of the perturbation velocities in the flowfield. These can, however, be related to the velocities at the profile surface in an approximated manner of various degrees of accuracy.

One example chosen by Nixon to point out the feature of the extended integral equation method is the supercritical flow